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A projection based approach to the Clebsch–Gordan multiplicity problem for compact semisimple Lie groups: II. Application to U(n)

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Abstract. Methods introduced in the preceding paper for resolving the multiplicity of irreducible subrepresentations occurring in the decomposition of the tensor product of two irreducible representations of a compact semisimple Lie group are illustrated by application to U(3), U(4) and general U(n). For U(3) we rederive very simply the known multiplicity structure for an irreducible tensor operator of fixed shift weight in terms of the decomposition of tensor product highest weight vectors into certain direct product states. We use the method to illustrate structural parallels between the Clebsch-Gordan problem for general U(n) and the U(3) case. Finally, we study in detail the multiplicity structure for a specific U(4) irreducible operator showing both the similarities and differences with U(3).

1. Introduction

The explicit decomposition of tensor product representations of the compact semisimple Lie groups, particularly the (special) unitary, orthogonal and symplectic groups, is a fundamental problem in the application of representation theory to the study of many-electron and many-nucleon systems. In the preceding paper (Edwards and Gould 1986, hereinafter referred to as I) we developed and extended a result of Parthasarathy *et al* (1967) which characterises multiplicities of irreducible components of tensor product representations simply in terms of Lie algebra action on weight spaces. We showed in general terms how this result may be used as a tool for an explicit resolution of the multiplicity in tensor product representations. In this paper we shall illustrate in detail how the tool may be applied to analyse products of irreducible representations of U(3) and U(4).

For any compact semisimple Lie group, the multiplicity of the irreducible component of highest weight $\lambda_i+\mu$ in the product representation $\lambda\otimes\mu$ varies in a characteristic way with μ if λ and λ_i are held fixed. For small μ the multiplicity is zero, while for large μ the multiplicity is equal to its maximal value, the dimension of the weight space $V_i(\lambda)$. The transition from zero to the maximal multiplicity occurs inside a narrow 'boundary layer' in the μ lattice where intermediate values of the multiplicity occur. In the case of the group U(3), this structure has been studied in detail (Lohe *et al* 1977). The results are illustrated in figure 1. Each point in the diagram is assigned three coordinates by orthogonal projection onto each of the three axes oriented at 120° to one another. The points for which each coordinate is an

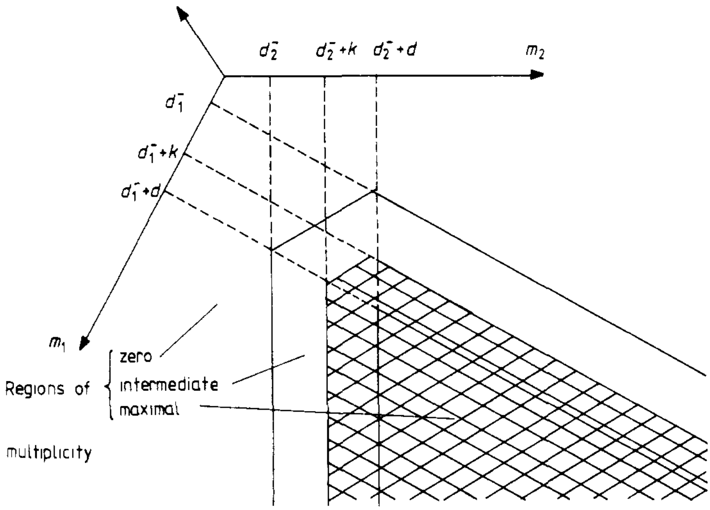

Figure 1. A 'pie-shaped' region for SU(3). The shaded region is the set of IR $\mu=(m_1, m_2)$ for fixed IR λ and shift weight λ_i for which the multiplicity of $\lambda_i+\mu$ in $\lambda\otimes\mu$ is not less than a fixed integer k.

integer form the weight lattice; the segment with non-negative components forms the lattice of possible highest weights. As μ varies over this segment, the level sets, for which the multiplicity of $\lambda_i+\mu$ in $\lambda\otimes\mu$ is some constant between 0 and dim $V_i(\lambda)$, form a set of lines which are the boundaries of a nested set of 'pie-shaped' regions.

In this paper we rederive these U(3) results very simply using the characterisation of highest weight vectors in tensor product representations introduced in I. We then examine the analogous multiplicity structures for U(4) and higher U(n), and show by taking an example from U(4) how the tools of I may be used to study them in detail.

2. Summary of needed results

In I we examined highest weight vectors (HWV) of irreducible subrepresentations of product representations $\lambda\otimes\mu$ of a compact semisimple Lie group G and focused on terms of the form $e\otimes e_+^\mu$ in their expansion in direct product states. We showed that a necessary and sufficient condition on a vector $e\in V(\lambda)$ of weight λ_i for a term $e\otimes e_+^\mu$ to occur as a summand in the expansion of a HWV is that it should be annihilated by each of the operators $y_k^{\langle\mu+\delta,\alpha_k\rangle}$. (Notation throughout this paper is as given in § 2 of I.)

It is convenient to express this annihilation property in terms of the spectral decomposition of e under the subalgebras of L isomorphic to sl(2) spanned by the sets $\{x_k, h_k, y_k\}$. The IR of sl(2) are simply labelled by their dimension. Fix λ and λ_i and write λ_{ik} for the kth component of λ. Each subalgebra $\mathrm{sl}(2)_k$ defines a nested sequence of subspaces of $V(\lambda)$, $\{0\}=V^k{}_0\subseteq V^k{}_1\subseteq V^k{}_2\subseteq \ldots\subseteq V^k{}_{m_{\max}}=V(\lambda)$, with $V^k{}_m$ spanned by the set of IR of $\mathrm{sl}(2)_k$ whose dimension is no greater than m. Let $W^k{}_m=V^k{}_m\cap V_i(\lambda)$. The nested sequence $\{0\}=W^k{}_0\subseteq W^k{}_1\subseteq W^k{}_2\subseteq\ldots\subseteq W^k{}_{m_{\max}}=V_i(\lambda)$ gives the $\mathrm{sl}(2)_k$ spectral properties of the weight space $V_i(\lambda)$. Write m_k for $\langle\mu,\alpha_k\rangle$. The condition $y_k^{m_k+1}e=0$ for $e\in V_i(\lambda)$ is equivalent to the condition $e\in W^k{}_{n_k}$ with $n_k=2m_k-\lambda_{ik}+1$. The condition that $e\otimes e_+^\mu$ appears in an expansion of a HWV in direct product states is now

$$e \in W^1{}_{n_1} \cap W^2{}_{n_2} \cap \ldots \cap W^l{}_{n_l}.$$

The study of multiplicity resolution for tensor products of G is in this way reduced to a study of the intersections of $\mathrm{sl}(2)_k$ submodules in single IR of G.

3. Multiplicity structure for U(3)

The $\mathrm{sl}(2)_k$ decompositions of IR of U(3) are easy to characterise because of the simplicity of the branching law for $\mathrm{U}(3) \supset \mathrm{U}(2)$. Fix a U(3) highest weight λ (defined with respect to the Cartan subalgebra spanned by $\{a^1{}_1, a^2{}_2, a^3{}_3\}$ where $a^i{}_j$ is the matrix with 1 in the (i, j)th position and zeros elsewhere). Let its components be $(\lambda_1, \lambda_2, \lambda_3)$. Let the components of a weight λ_i occurring in $V(\lambda)$ be $(\lambda_{i1}, \lambda_{i2}, \lambda_{i3})$. The weight space $V_i(\lambda)$ has two different bases, one of Gel'fand vectors arising from the chain $\mathrm{U}(3) \supset \mathrm{U}(2)_1 \supset \mathrm{U}(1)_1$ corresponding to the algebra $\mathrm{sl}(2)_1$ spanned by $\{a^1{}_2, a^1{}_1 - a^2{}_2, a^2{}_1\}$, and the other from the chain $\mathrm{U}(3) \supset \mathrm{U}(2)_2 \supset \mathrm{U}(1)_2$ corresponding to the algebra $\mathrm{sl}(2)_2$ spanned by $\{a^2{}_3, a^2{}_2 - a^3{}_3, a^3{}_2\}$. Let $\mathrm{d} = \dim V_i(\lambda)$. With respect to the first subgroup chain, $V_i(\lambda)$ has a Gel'fand basis $\{\xi_1, \xi_2, \ldots, \xi_d\}$ where ξ_m has the Gel'fand pattern

$$\begin{pmatrix} \lambda_1 \quad \lambda_2 \quad \lambda_3 \\ \rho_1 + m \quad \sigma_1 - m \\ \lambda_{i1} \end{pmatrix}$$

with ρ_1 and σ_1 defined by the conditions

$$\rho_1 + \sigma_1 = \lambda_{i1} + \lambda_{i2}$$

$$\rho_1 + 1 = \max\{\lambda_{i1}, \lambda_2, \lambda_{i2}, \lambda_{i1} + \lambda_{i2} - \lambda_3\}.$$

Similarly with respect to the second chain, $V_i(\lambda)$ has a Gel'fand basis $\{\eta_1, \eta_2, \ldots, \eta_d\}$ where η_m has the Gel'fand pattern

$$\begin{pmatrix} \lambda_1 \quad \lambda_2 \quad \lambda_3 \\ \rho_2 + m \quad \sigma_2 - m \\ \lambda_{i2} \end{pmatrix}$$

$$\rho_2 + \sigma_2 = \lambda_{i2} + \lambda_{i3}$$

$$\rho_2 + 1 = \max\{\lambda_{i2}, \lambda_2, \lambda_{i3}, \lambda_{i2} + \lambda_{i3} - \lambda_3\}.$$

The spectral decomposition of $V_i(\lambda)$ under $\mathrm{sl}(2)_1$ (resp $\mathrm{sl}(2)_2$) yields the d one-dimensional subspaces $\{\mathbb{C}\xi_1, \ldots, \mathbb{C}\xi_d\}$ (resp $\{\mathbb{C}\eta_1, \ldots, \mathbb{C}\eta_d\}$). For simplicity we shall denote the 'cumulative' spectral subspaces $W^k{}_m$ defined in § 2 by

$$V^k{}_m \equiv W^k_{\rho_k - \sigma_k + 2m - 1}$$

giving the ascending chains

$$\{0\} = V^1{}_0 \subset V^1{}_1 \subset \ldots \subset V^1{}_d = V_i(\lambda)$$

$$\{0\} = V^2{}_0 \subset V^2{}_1 \subset \ldots \subset V^2{}_d = V_i(\lambda).$$

$V^1{}_m$ is the linear span of $\{\xi_1, \ldots, \xi_m\}$ and $V^2{}_m$ is the linear span of $\{\eta_1, \ldots, \eta_m\}$.

The multiplicity structure for U(3) (shown in figure 1) is an immediate consequence of the following fact:

$$\dim(V^1{}_r \cap V^2{}_s) = \max(r + s - d, 0).$$

This will be proved below. The multiplicity of $\lambda_i + \mu$ in $\lambda \otimes \mu$ is $\dim(V^1{}_r \cap V^2{}_s)$ with r

given by

$$r=\begin{cases} d & \text{if } \lambda_1-d_1\geqslant d\\ \lambda_1-d_1 & \text{if } 0\leqslant\lambda_1-d_1\leqslant d\\ 0 & \text{if } \lambda_1-d_1\leqslant 0\end{cases}$$

where $d_1=\rho_1-\lambda_{i1}=\max(0,\lambda_2-\lambda_{i1},-\lambda_{i1},\lambda_{i2}-\lambda_2)$, and s given by an analogous formula in terms of d_2. The set of IR μ of G for which the multiplicity of $\lambda_i+\mu$ in $\lambda\otimes\mu$ is at least $m\in\{0,1,\ldots,d\}$ is seen to be the 'pie-shaped' region

$$P_m\equiv\{\mu\in\Delta_+|\mu_1\geqslant d_1+m,\ \mu_2\geqslant d_2+m,\ \mu_1+\mu_2\geqslant d_1+d_2+d+k\}.$$

The assertion $\dim(V^1{}_r\cap V^2{}_s)=\max(r+s-d,0)$ follows directly by simple dimension counting from the statement $\dim(V^1{}_r\cap V^2{}_s)=0$ if $r+s=d$, since $\dim V^1{}_r=r$ and $\dim V^2{}_s=s$. We can prove the latter statement by induction over r as follows. It is obvious for $r=0$. Suppose $V^1{}_m\cap V^2{}_{d-m}=\{0\}$. Let C be the second-order Casimir operator for $U(2)_1$ and observe that the $U(2)_2$ shift properties of the $U(2)_1$ generators imply that $CV^2{}_{d-m-1}\subset V^2{}_{d-m}$. All the eigenspaces of C in $V_i(\lambda)$ are one-dimensional since C separates different IR of $sl(2)_1$, so if c_{m+1} is the eigenvalue of C for ξ_{m+1} and $e\in V_i(\lambda)$

$$(C-c_{m+1})e=0\Leftrightarrow e\in\mathbb{C}\xi_{m+1}.$$

Hence $V^1{}_{m+1}\cap V^2{}_{d-m-1}\subseteq\mathbb{C}\xi_{m+1}$ since

$$\begin{aligned}(C-c_{m+1})(V^1{}_{m+1}\cap V^2{}_{d-m-1})&\subseteq V^1{}_m\cap V^2{}_{d-m}\\ &=\{0\}\ \text{by assumption.}\end{aligned}$$

From the known matrix elements of the generators of $U(2)_2$ in the $U(2)_1$ Gel'fand basis, it is easy to check that all the Gel'fand vectors ξ_j have non-zero maximal spectral component under $U(2)_2$: in other words

$$\langle\xi_{m+1},\eta_d\rangle\neq 0.$$

Hence, for $m+1<d$, $\xi_{m+1}\notin V^2{}_{d-m-1}$ forcing $V^1{}_{m+1}\cap V^2{}_{d-m-1}=\{0\}$ as required. For $m+1=d$, $V^2{}_{d-m-1}=V^2{}_0=\{0\}$ completing the proof.

4. Multiplicity structure for U(4) and higher U(n)

U(3) has an exceptionally simple multiplicity structure which is embodied in the formula $\dim(V^1{}_r\cap V^2{}_s)=\max(r+s-d,0)$. Although Clebsch-Gordan decompositions for higher U(n) are substantially more complicated than for U(3), a number of general structural features remain. In particular the zero, maximal and intermediate multiplicity regions give rise to higher-dimensional 'pie-shaped' regions, with the region of intermediate multiplicity forming a narrow boundary layer between the zero and maximal multiplicity regions. Some properties of this boundary layer can be gleaned from the results of I.

Each $sl(2)_k$ subalgebra of L gives rise to a nested sequence of subspaces of a weight space $V_i(\lambda)$

$$\{0\}=V^k{}_0\subset V^k{}_1\subset\ldots\subset V^k{}_d=V_i(\lambda)$$

in just the same way as constructed above for U(3). The multiplicity structure of the Clebsch-Gordan problem is embedded in the l-fold intersections of these subspaces.

The problem of studying these intersections in the case of $U(n)$ is in principle relatively straightforward since $sl(2)_k$-adapted subspaces can readily be constructed and enumerated using Gel'fand bases. The details of three- or more-fold intersections rapidly become complex as n increases. However, for a vast majority of cases it is only necessary to study one- and two-fold intersections, corresponding to situations, for fixed λ and shift weight λ_i, where no more than two of the components of μ lie in the intermediate multiplicity region. In such cases the other components of μ either yield a zero subspace (producing a zero subspace for the intersection of all the $V^k{}_m$) or else a maximal subspace (i.e. the whole of $V_i(\lambda)$—which can be disregarded in determining intersections). This will be verified in detail below with an example from U(4).

When all but one of the components of μ yield maximal subspaces, the multiplicity structure (for arbitrary compact semisimple groups) is precisely analogous to the structure along the long sides of the U(3) 'pie-shaped' regions. Suppose the exceptional component is k. Then the multiplicity increases monotonically as $\langle \mu, \alpha_k \rangle$ moves through the region of intermediate multiplicity from zero to dim $V_i(\lambda)$, jumping in steps given by dim $V^k{}_1$, dim $V^k{}_2$, etc. The multidimensional 'pie-shaped' regions for a Lie algebra of rank l will have l such $(l-1)$-dimensional sides.

When all but two of the components of μ yield maximal subspaces (corresponding in the U(3) case to the truncated top apex of the pie slice), there are essentially only two possibilities. Either the two relevant $sl(2)_k$ subalgebras will commute or they will fail to commute. When they commute, the intersections between the spectral subspaces of the two $sl(2)_k$ subalgebras are simply determined for $U(n)$ using Gel'fand bases. When they fail to commute, the two subalgebras generate together an algebra isomorphic to sl(3) and the problem is identical to that for U(3): the same truncated pie slices will appear in cross section. It is worth noting that spectral enumeration via Gel'fand bases completely determines n-fold $sl(2)_k$ intersections whenever all n subalgebras mutually commute. We now illustrate with an example.

Take $G = U(4)$, $\lambda = (11, 8, 4, 1)$ and $\lambda_i = (5, 10, 6, 3)$. With the matrices $a^i{}_j$ (1 in the (i, j) position, zeros elsewhere) as a basis for L, the three sl(2) subalgebras can be taken as the spans of $\{a^1{}_2, a^1{}_1 - a^2{}_2, a^2{}_1\}$, $\{a^2{}_3, a^2{}_2 - a^3{}_3, a^3{}_2\}$ and $\{a^3{}_4, a^3{}_3 - a^4{}_4, a^4{}_3\}$. Denote them $sl(2)_1$, $sl(2)_2$ and $sl(2)_3$, and the three corresponding U(2) subgroups $U(2)_1$, $U(2)_2$ and $U(2)_3$. Enumerating Gel'fand basis weight vectors with respect to the appropriate subgroup embeddings yields the following spectral decomposition of the eight-dimensional (5, 10, 6, 3) weight space:

$$U(2)_1: 3\times(11,4)\oplus 5\times(10,5)$$

$$U(2)_2: 3\times(11,5)\oplus 5\times(10,6)$$

$$U(2)_3: 2\times(8,1)\oplus 3\times(7,2)\oplus 3\times(6,3).$$

Hence

$$\dim V^1{}_1 = 5 \qquad \dim V^1{}_2 = 8$$

$$\dim V^2{}_1 = 5 \qquad \dim V^2{}_2 = 8$$

$$\dim V^3{}_1 = 3 \qquad \dim V^3{}_2 = 6 \qquad \dim V^3{}_3 = 8.$$

With $\mu = (m_1, m_2, m_3, m)$, the intermediate multiplicity regions occur for

$$m_1 - m_2 = 0 \qquad m_2 - m_3 = 4 \qquad m_3 - m_4 = 3, 4.$$

The dimensions of the intersections $V^1{}_r \cap V^2{}_s \cap V^3{}_t$ for $r = 1, 2$, $s = 1, 2$ and $t = 1, 2, 3$

form the following $2\times3\times3$ matrix of integers:

$$
r=1:\quad \begin{array}{r|lll} 2 & 2 & 4 & 5 \\ s:\ 1 & 0 & 1 & 2 \\ \hline t: & 1 & 2 & 3 \end{array}
\qquad\qquad
r=2:\quad \begin{array}{r|lll} 2 & 3 & 6 & 8 \\ s:\ 1 & 1 & 3 & 5 \\ \hline t: & 1 & 2 & 3 \end{array}
$$

This gives the details of the truncated apex of a wedge-shaped multiplicity diagram.

In all but the two cases $(r, s, t)=(1,1,1)$ and $(1,1,2)$, one of the three subspaces ${V^1}_r$, ${V^2}_s$ and ${V^3}_t$ was maximal in the above calculation so that only a pairwise intersection needed to be considered. The cases ${V^1}_r\cap {V^2}_s$ (with ${V^3}_t$ maximal) and ${V^2}_s\cap {V^3}_t$ (with ${V^1}_r$ maximal) were evaluated by enumerating Gel'fand bases reduced with respect to the U(3) subgroups acting on the indices $\{1,2,3\}$ and $\{2,3,4\}$ respectively. This reduced the intersection determination to a series of U(3) cases, which were solved in § 3. The case ${V^1}_r\cap {V^3}_t$ (with ${V^2}_s$ maximal) is evaluated by enumerating Gel'fand bases reduced with respect to the $\{1,2,3\}$ U(3) and the $\{1,2\}$ U(2) for the weights $(5,10,6,3)$, $(5,10,7,2)$ and $(5,10,8,1)$: this yields a spectral decomposition of the $(5,10,6,3)$ weight space under $U(2)_1\times U(2)_3$ and the pairwise intersection spaces follow directly. The two cases involving three-way intersections had to be solved by considering known matrix elements of the $U(2)_3$ generators acting on vectors in the pairwise intersection ${V^1}_r\cap {V^2}_s$.

The apex of the multiplicity diagram can be seen from this example to have a structure sufficiently complex that a general detailed description of this region for arbitrary $U(n)$ seems to be out of the question. However the example also makes it clear that simple tools of spectral analysis such as Gel'fand basis enumeration can go a long way towards a complete solution in individual cases.

5. Conclusion

For brevity we have confined ourselves in this paper to giving simplified derivations of the multiplicity diagrams for the various cases considered. This is only the first of a long series of steps necessary in producing complete algorithms for the computation of Clebsch–Gordan coefficients for $U(n)$ using the methods of I. The next stage will include considering the form of the $U(n):U(n-1)$ reduced coefficients implied by these methods, and to analyse and exploit the many Weyl group symmetries present. The characterisation of highest weight vectors in tensor product representations via their components of the form $e\otimes e^{\mu}_{+}$ lends itself directly to further analysis via subgroup chains and Weyl symmetries. This is the subject of continuing research.

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